ON DERIVATIONS OF FINITE INDEX CHAIN CONDITIONS, DIMENSIONS AND RADICALS

BY

PIOTR GRZESZCZUK

Institute of Mathematics, University of Warsaw, Bialystok Division, Akademicka 2, 15–267 Bialystok, Poland

ABSTRACT

Let d be a derivation of an associative ring R, and let M be a left R-module with d-derivation D of finite index. It is shown that M satisfies any of a class of conditions (including ACC, DCC, uniform, Gabriel, Krull dimension) if and only if it satisfies the same condition with respect to D-invariant submodules. If in addition $1/w! \in R$, where w denotes the index of d, then D-simple R-modules are completely reducible. Relationships between the Jacobson and the D-invariant Jacobson radicals of M are investigated.

0. Introduction

Let R be an associative ring with unity and let $d\colon R \to R$ be a derivation of R, i.e., $d(a+b)=d(a)+d(b)$ and $d(ab)=d(a)b+ad(b)$ for all $a, b \in R$. A left ideal I of R is a d-ideal provided $d(I) \subseteq I$. We say that R is d-Noetherian if every ascending chain of left d-ideals of R stabilizes. The aim of this paper is to study relations between some properties of R and their d-counterparts. It appears that, although generally one can not expect too much, for enough good derivations the relations are rather close. Many of the proofs need information about factors R/I, where I is a left d-ideal of R. The mapping $\bar{d}\colon R/I \to R/I$, induced by d, satisfies for every $r \in R$, $m \in R/I$, $\bar{d}(rm)=d(r)m+r\bar{d}(m)$. More generally, an endomorphism D of the additive group of a left R-module M is said to be d-derivation of M if $D(rm)=d(r)m+rD(m)$ for every $r \in R$, $m \in M$. A submodule N of M is a D-submodule provided $D(N) \subseteq N$. Most of the results of this paper concern R-modules with d-derivations. We also need some auxiliary results on partially ordered sets. It is not surprising because the properties studied and their D-counterparts can be expressed as properties of

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the lattice $L(M)$ of submodules of M and the lattice $L_D(M)$ of D-submodules of M.

1. Preliminaries

In this section we state some properties of a left R-module M with d-derivation D. Extending the product rule in the definition of D we obtain the Leibniz formula:

$$D^n(rm) = \sum_{j=0}^{n} \binom{n}{j} d^{n-j}(r) D^j(m). \tag{1.1}$$

Let N be a D-submodule of M. Putting, $\bar{D}(m+N) = D(m) + N$ we obtain a d-derivation on M/N which is said to be induced by D. For every submodule N of M let

$$N^* = \sum_{j=0}^{\infty} D^j(N) \left(N_* = \bigcap_{j=0}^{\infty} D^{-j}(N) \right), \quad \text{where } D^{-j}(N) = \{x \in M \mid D^j(x) \in N\}.$$

Clearly N^* (N_*) is the smallest (largest) D-submodule of M containing N (contained in N).

1.2. Lemma. *For every submodule N of M, the R-modules $N + D(N)/N$ and $N/N \cap D^{-1}(N)$ are isomorphic. In particular $L(N + D(N)/N) \approx L(N/N \cap D^{-1}(N))$.*

Proof. It is easy to see that the map $f\colon N \to N + D(N)/N$ given by $f(n) = D(n) + N$ is an epimorphism of R-modules and $\ker f = N \cap D^{-1}(N)$.

The following proposition is straightforward.

1.3. Proposition. *Given an integer $v \geqq 1$ the following conditions are equivalent:*

(i) *for every $m \in M$, $D^v(m) \in \sum_{j=0}^{v-1} RD^j(m)$;*
(ii) *for every submodule N of M, $N^* = \sum_{j=0}^{v-1} D^j(N)$;*
(iii) *for every submodule N of M, $N_* = \bigcap_{j=0}^{v-1} D^{-j}(N)$.*

1.4. Definition. We say that the d-derivation D has finite index if there exists an integer $v \geqq 1$ satisfying Proposition 1.3. The index of D is defined as the smallest number v from 1.3.

Throughout the paper D denotes a d-derivation of M of finite index, generally denoted by v. If $(X, \leqq)$ is a partially ordered set, then the interval $\{x \in X \mid a \leqq x \leqq b\}$ is written $[a, b]_X$ or $[a, b]$ if the context is clear.

Let $b, B: L(M) \to L(M)$ be defined by $b(N) = N \cap D^{-1}(N)$, $B(N) = N + D(N)$. Clearly b and B have the following properties:

1.5. For every submodule N of M

(i) $b(N) \subseteq N \subseteq B(N)$, $Bb(N) \subseteq N \subseteq bB(N)$;

(ii) $[b(N), N] \approx [N, B(N)]$;

(iii) $N_* = b^{v-1}(N)$, $N^* = B^{v-1}(N)$.

1.6. If in addition $N_* = 0$, then

$$M \supseteq N \supseteq b(N) \supseteq b^2(N) \supseteq \cdots \supseteq b^{v-1}(N) = 0$$

and for every $j = 1, 2, \ldots, v-1$

$$[b^j(N), b^{j-1}(N)] \approx [b^{j-1}(N), Bb^{j-1}(N)] \subseteq [b^{j-1}(N), b^{j-2}(N)],$$

where $b^0(N) = N$ and $b^{-1}(N) = B(N)$.

Now we will prove a result analogous to Lemma 1.3 of [1].

1.7. Lemma. *There exists a submodule N of M maximal with respect to $N_* = 0$.*

Proof. By Zorn's lemma it suffices to show that if $\{N_\alpha \mid \alpha \in A\}$ is a chain of submodules of M such that $(N_\alpha)_* = 0$ for all $\alpha \in A$, then $(\bigcup N_\alpha)_* = 0$. If $(\bigcup N_\alpha)_* \neq 0$ then $0 \neq (Rm)^* = \sum_{j=0}^{v-1} RD^j(m) \subseteq \bigcup N_\alpha$ for some $m \in \bigcup N_\alpha$. Since $D^j(m) \in \bigcup N_\alpha$, one has $D^j(m) \in N_{\alpha(j)}$ for some $\alpha(j) \in A$. Choosing $\beta = \max \alpha(j)$, we obtain that $0 \neq (Rm)^* \subseteq N_\beta$, a contradiction to $(N_\beta)_* = 0$.

2. Some results on partially ordered sets

In this section we prove some general results about partially ordered sets. We will use them for examining the relation between the uniform dimension and the D-uniform dimension of factors of a module M.

Let $\mathcal{K}$ be a class of partially ordered sets with 0 and 1, which is closed with respect to isomorphisms (of partially ordered sets). Moreover we assume that if $X \in \mathcal{K}$, then for every $x, y \in X$ with $x \leqq y$, $[x, y] \in \mathcal{K}$. Now let Ω be a property of partially ordered sets. If a partially ordered set $(X, \leqq)$ has property Ω, then we write $X \in \Omega$.

2.1. Definition. The property Ω is said to be upper hereditary on $\mathcal{K}$ if Ω satisfies the following three conditions:

(i) if $X \approx Y$ and $X \in \Omega$, then $Y \in \Omega$;

(ii) if $x, y, z \in X \in \mathcal{K}$ $(x \leqq y \leqq z)$ and $[x, y] \in \Omega$ then $[x, z] \in \Omega$;

(iii) if $X \in \Omega$ then for every $x \in X$ $[0, x] \in \Omega$ or $[x, 1] \in \Omega$.

2.2. EXAMPLE. Let $\mathcal{M}$ be the class of all modular lattices with 0 and 1. The following properties are upper hereditary on $\mathcal{M}$:

(a) $L \in \Omega$ if and only if L is not Noetherian (L is not Artinian, L has infinite length).

(b) $L \in \Omega$ if and only if $\text{u.dim}\, L = \infty$, where $\text{u.dim}\, L$ denotes the uniform dimension of L. (For the basic properties of the uniform dimension of modular lattices we refer to [5].)

2.3. PROPOSITION. *Let Ω be an upper hereditary property on the class $\mathcal{K}$ and let $X \in \mathcal{K}$. Suppose that there exist order preserving maps $b, B: X \to X$ such that for every $x \in X$*

(i) $b(x) \leqq x \leqq B(x)$ *and* $Bb(x) \leqq x \leqq bB(x)$,

(ii) $[b(x), x] \approx [x, B(x)]$.

If for some $x_0 \in X$, $[x_0, 1] \in \Omega$, then for every integer $k \geqq 0$ there exists an element $x \in X$ such that $[b^k(x), 1] \in \Omega$ or $[B^k(x), 1] \in \Omega$.

The proof of this proposition requires some preliminary lemmas.

2.4. LEMMA. *If $k \geqq l \geqq 0$, then $b^l B^l b^k = b^k$ and $B^l b^l B^k = B^k$.*

PROOF. Let $x \in X$. Since b and B are order preserving maps and for every $y \in X$, $Bb(y) \leqq y \leqq bB(y)$, we obtain

$$b^l B^l b^k(x) = b^{l-1}(bB(B^{l-1}b^k(x))) \geqq b^{l-1}B^{l-1}b^k(x) \geqq \cdots \geqq b^k(x).$$

On the other hand

$$b^l B^l b^k(x) = b^l B^{l-1}(Bb(b^{k-1}(x))) \leqq b^l B^{l-1} b^{k-1}(x) \leqq \cdots \leqq b^l b^{k-l}(x) = b^k(x).$$

Hence $b^l B^l b^k = b^k$. The proof of the second equality is analogous.

2.5. LEMMA. *For every $z \in X$ and $1 \leqq m < n$, if $[b^m B^n(z), b^{m-1}B^n(z)] \in \Omega$ then $[b^{m+1}B^n(z), b^m B^n(z)] \in \Omega$.*

PROOF. The proof is by induction on n. Let $n = 2$ and let $[bB^2(z), B^2(z)] \in \Omega$. By 2.4 we have

$$[bB^2(z), B^2(z)] = [bB^2(z), BbB^2(z)] \approx [b^2B^2(z), bB^2(z)],$$

so $[b^2B^2(z), bB^2(z)] \in \Omega$.

Suppose that the lemma is true for all $j < n$. Now, let $m < n$ and let $[b^m B^n(z), b^{m-1}B^n(z)] \in \Omega$. Consider the chain

$$b^m B^n(z) \leqq Bb^m B^n(z) \leqq B^2 b^m B^n(z) \leqq \cdots \leqq B^m b^m B^n(z) = B^n(z).$$

Since $b^{m-1}B^n(z) \leqq B^n(z)$, by upper hereditary, we obtain that $[b^m B^n(z), B^n(z)] \in \Omega$. Hence for some $j < m < n$

$$(*) \qquad [B^j b^m B^n(z), B^{j+1} b^m B^n(z)] \in \Omega.$$

Choose j minimal satisfying (*). We claim that $j \leqq 1$. Indeed, if $j \geqq 2$ then

$$\Omega \ni [B^j b^m B^n(z), B^{j+1} b^m B^n(z)] \approx [bB^j b^m B^n(z), B^j b^m B^n(z)].$$

Hence for $w = b^m B^n(z)$, $[bB^j(w), B^j(w)] \in \Omega$. The induction assumption gives

$$\Omega \ni [b^j B^j(w), b^{j-1} B^j(w)] = [b^j B^j b^m B^n(z), b^{j-1} B^j b^m B^n(z)].$$

Since $b^j B^j b^m = b^m$ and $b^{j-1} B^j b^m B^n(z) \leqq B^j b^m B^n(z)$, we have $[b^m B^n(z), B^j b^m B^n(z)] \in \Omega$. This contradicts minimality of j and proves the claim. Hence $0 \leqq j \leqq 1$. Let us observe that

$$[Bb^m B^n(z), B^2 b^m B^n(z)] \approx [bBb^m B^n(z), Bb^m B^n(z)]$$
$$= [b^m B^n(z), Bb^m B^n(z)] \approx [b^{m+1} B^n(z), b^m B^n(z)].$$

Therefore $j = 0$ and $[b^{m+1} B^n(z), b^m B^n(z)] \in \Omega$.

Proof of Proposition 2.3. We proceed by induction, the case $k = 0$ being true from the assumption. Now let $k \geqq 0$. We have two possibilities: $[b^k(x), 1] \in \Omega$ or $[B^k(x), 1] \in \Omega$. The second case reduces to the first one. Indeed, if $[B^k(x), 1] \in \Omega$ then $[B^k(x), B^{k+1}(x)] \in \Omega$ or $[B^{k+1}(x), 1] \in \Omega$. We have $[B^k(x), B^{k+1}(x)] \approx [bB^k(x), B^k(x)]$, so if $[B^k(x), B^{k+1}(x)] \in \Omega$ then by Lemma 2.5 $[b^k B^k(x), b^{k-1} B^k(x)] \in \Omega$ and by upper hereditary $[b^k(z), 1] \in \Omega$, for $z = B^k(x)$.

Now let $[b^k(x), 1] \in \Omega$. Consider the chain

$$b^k(x) \leqq Bb^k(x) \leqq \cdots \leqq B^k b^k(x) \leqq B^{k+1} b^k(x) \leqq \cdots \leqq 1.$$

If $[B^{k+1} b^k(x), 1] \in \Omega$ then $[B^{k+1}(z), 1] \in \Omega$ for $z = b^k(x)$. Hence we may assume that $[B^{k+1} b^k(x), 1] \notin \Omega$, so that $[b^k(x), B^{k+1} b^k(x)] \in \Omega$. By upper hereditary there is $j \leqq k$ minimal with respect to $[B^j b^k(x), B^{j+1} b^k(x)] \in \Omega$. We claim that $j \leqq 1$. Indeed, if $j \geqq 2$ then

$$\Omega \ni [B^j b^k(x), B^{j+1} b^k(x)] \approx [bB^j b^k(x), B^j b^k(x)],$$

so by Lemmas 2.5 and 2.4 $\Omega \ni [b^j B^j b^k(x), b^{j-1} B^j b^k(x)] = [b^k(x), b^{j-1} B^j b^k(x)]$. Applying the same argument as in the proof of Lemma 2.5 we obtain a contradiction with minimality of j. Therefore $0 \leqq j \leqq 1$ and $[b^{k+1}(x), 1] \in \Omega$. This ends the proof.

As an immediate consequence of Proposition 2.3 and Example 2.2(b) we obtain the following

2.6. COROLLARY. *Let X be a modular lattice with 0, 1 and let Y be its subset. If there exist order preserving maps $b, B: X \to X$ and an integer $n \geqq 1$ such that for every $x \in X$*

(i) $b(x) \leqq x \leqq B(x)$ *and* $Bb(x) \leqq x \leqq bB(x)$,

(ii) $[b(x), x] \approx [x, B(x)]$,

(iii) $b^n(x) \in Y$ *and* $B^n(x) \in Y$,

then for every $x \in X$ $\mathrm{u.dim}[x, 1] < \infty$ if and only if $\mathrm{u.dim}[y, 1] < \infty$ for every $y \in Y$.

3. Chain conditions and dimensions

In this section we study relations between chain conditions and dimensions of M and their D-counterparts. By D-$l(M)$ (D-$\mathrm{u.dim}(M)$, D-$\mathrm{G\,dim}(M)$, D-$\mathrm{K\,dim}(M)$) we denote the length (uniform, Gabriel, Krull dimensions, respectively) of M with respect to D-submodules.

The statements (b) and (c) of the following theorem are related respectively to Theorem 2.1 of [2] and Lemma 1.4 of [1] and they will be proved in almost the same way.

3.1. THEOREM. *Let M be a left R-module with a d-derivation D of finite index v. Then*

(a) D-$l(M) \leqq l(M) \leqq v \cdot D$-$l(M)$;

(b) *M is Noetherian if and only if M is D-Noetherian*;

(c) D-$\mathrm{u.dim}(M) \leqq \mathrm{u.dim}(M) \leqq v \cdot D$-$\mathrm{u.dim}(M)$;

(d) *for every submodule N of M $\mathrm{u.dim}\, M/N < \infty$ if and only if for every D-submodule K of M $\bar{D}$-$\mathrm{u.dim}\, M/K < \infty$, where $\bar{D}$ is an induced by D d-derivation on M/K*;

(e) *M is Artinian if and only if M is D-Artinian.*

PROOF. (a) Clearly it suffices to consider the case when M is D-simple. By 1.7 we can take a submodule N of M, maximal with respect to $N_* = 0$. Since M has no proper D-submodules, M/N is a simple module. By 1.6, for every $j = 0, 1, \ldots, v-1$, the module $b^j(N)/b^{j+1}(N)$ is simple or it is equal to zero. Hence $l(M) \leqq v$.

(b) Suppose that M is D-Noetherian. Using Noetherian induction we can assume that for every non-zero D-submodule X of M the module M/X is Noetherian. Now let N be a submodule of M maximal with respect to $N_* = 0$. Let $N \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subseteq M$ be an ascending chain of submodules of M. Since $N \subsetneq X_1$, $0 \neq (X_1)_* \subseteq X_1$. By our assumption $M/(X_1)_*$ is Noetherian, so the chain $X_1 \subsetneq X_2 \subsetneq \cdots$ terminates. Therefore the module M/N is Noetherian and by 1.6 so is the module M.

The converse is clear.

(c) Let D-u.dim$(M) = k < \infty$. By 1.6 it suffices to show that u.dim$(M/N) \leqq k$, where N is a submodule of M maximal with respect to $N_* = 0$. Let $N_1, N_2, \ldots, N_m$ be submodules of N strictly containing N whose sum is direct modulo N. Clearly for every $j = 1, 2, \ldots, m$ $(N_j)_* \neq 0$ and

$$(N_j)_* \cap \sum_{i \neq j} (N_i)_* \subseteq (N_j)_* \cap \Big(\sum_{i \neq j} N_i\Big)_* = N_* = 0.$$

Therefore $m \leqq k$.

(d) It is an immediate consequence of 2.6 and 3.1(c).

(e) Assume that M is D-Artinian. We claim that for every submodule X of M there exists a submodule Y of M minimal over X. Obviously there exists a D-submodule K of M minimal over X_* as a D-submodule. Clearly $\bar{D}$-$l(K/X_*) = 1$, where $\bar{D}$ is the induced by D d-derivation of K/X_*. Since $X_* \subseteq K \cap X$ and $0 \neq K/K \cap X \approx K + X/X$, by 3.1(a), we obtain that $l(K + X/X) \leqq v$. This proves the claim.

Now it is easy to see that for submodules $X \subseteq Z$ of M there exists a submodule $Y \subseteq Z$ minimal over X.

The above remarks and 3.1(d) imply that for every submodule X of M the socle Soc(M/X) is a finite direct sum of simple submodules of M/X and that Soc(M/X) is an essential submodule of M/X.

Now let $M_1 \supseteq M_2 \supseteq \cdots$ be a descending chain of submodules of M and let $X = \bigcap_{n=1}^{\infty} M_n$. Since Soc$(M/X)$ is essential in M/X and it has finite length, the chain terminates.

Now we will examine the relationship between Gabriel, G dim(M), and D-Gabriel, D-G dim(M), dimensions of M. For the definitions and the basic properties of this dimension we refer to [6].

3.2. THEOREM. G dim$(M) = D$-G dim(M) *if either side exists.*

PROOF. We prove separately

(a) if D-G dim$(M) = \alpha$ then G dim$(M) \leqq \alpha$;

(b) if G dim$(M) = \alpha$ then D-G dim$(M) \leqq \alpha$.

Suppose that (a) does not hold. Let α be the smallest ordinal number such that for some module M, D-G dim$(M) = \alpha$ and G dim$(M) \nleqq \alpha$. Let $T = \Sigma\{N \subseteq M \mid \text{G dim}(N) \leqq \alpha\}$. Clearly G dim$(T) \leqq \alpha$ and by 1.2, $T = T^*$. Hence we can assume that $T = 0$. Obviously we can assume that M is α-D-simple. Let N be a submodule of M maximal with respect to $N_* = 0$. For every submodule X strictly containing N we have $\bar{D}$-G dim $M/X_* < \alpha$. Now minimality of α gives G dim$(M/X) \leqq$ G dim$(M/X_*) < \alpha$. Therefore G dim$(M/N) \leqq \alpha$ and, by 1.6, G dim$(M) \leqq \alpha$, a contradiction.

Now we prove (b). The proof is by induction on α. Let T be the largest D-submodule of M such that D-G dim$(T) \leqq \alpha$. We claim that $T = M$. If not, we may assume that $T = 0$ and $M \neq 0$. In this case, by the induction assumption and by 1.2, M contains no non-zero submodules X with G dim$(X) < \alpha$. Now since G dim$(M) = \alpha$, there exists an α-simple submodule N.

Let us define the relation $\sim$ $(\sim_D)$ on the lattice $L(M)$ $(L_D(M))$ putting $X \sim Y$ $(X \sim_D Y)$ if and only if

$$\text{G dim}(X + Y/X \cap Y) < \alpha \qquad (\bar{D}\text{-G dim}(X + Y/X \cap Y) < \alpha).$$

It is easy to check that $\sim$ and $\sim_D$ are congruence relations on $L(M)$ and $L_D(M)$ respectively. Now if X and Y are D-submodules of M, and G dim$(X + Y/X \cap Y) < \alpha$ then by the induction assumption $\bar{D}$-Gdim$(X + Y/X \cap Y) < \alpha$. This shows that the restriction of $\sim$ to $L_D(M)$ is equal to $\sim_D$. Hence $L_D(M)/\sim_D$ can be treated as a sublattice of $L(M)/\sim$. Obviously the congruence class $[N]$ containing N is an atom in the quotient lattice $L(M)/\sim$. As an immediate consequence of 1.2 we obtain that the interval $[[0], [N^*]]_{L(M)/\sim}$ has a finite length. Hence also the interval $[[0], [N^*]]_{L_D(M)/\sim_D}$ has a finite length. Therefore $L_D(M)/\sim_D$ contains an atom $[K]$. This implies that for every non-zero D-submodule X of K, D-G dim$(X) < \alpha$ or $\bar{D}$-G dim$(K/X) < \alpha$. Since $T = 0$, D-G dim$(X) \nleqq \alpha$. This implies that K is α-D-simple, so D-G dim$(K) = \alpha$. Using again the assumption that $T = 0$ we obtain a contradiction.

The proof of Theorem is complete.

Now we can prove the result on Krull dimension corresponding to 3.2. It generalizes 3.1(e). We use Gordon and Robson's result [3], which asserts that a module has Krull dimension if and only if it has Gabriel dimension and all its factors have finite uniform dimension.

3.3. THEOREM. K dim$(M) = D$-K dim(M) *if either side exists.*

PROOF. Clearly it suffices to show that if D-K dim$(M) = \alpha$ then K dim$(M) \leqq$

α. The quoted Gordon–Robson's result, 3.1(d) and 3.2 prove that $\mathrm{K}\dim(M)$ exists. By Corollary 4.2 of [4], M contains a largest submodule T with $\mathrm{K}\dim(T) \leqq \alpha$. Using 1.2 we obtain that $T = T^*$. Now passing if necessary to the factor module M/T we can assume that $T = 0$. Using the induction, we can also assume that M is α-D-critical. Now let N be a submodule of M maximal with respect to $N_* = 0$. If $X \supsetneq N$, then $X_* \neq 0$ and by the induction assumption $\mathrm{K}\dim(M/X) \leqq \mathrm{K}\dim(M/X_*) < \alpha$. Hence $\mathrm{K}\dim(M/N) \leqq \alpha$. Applying 1.6 we obtain that $\mathrm{K}\dim(M) \leqq \alpha$, a contradiction. Therefore $T = M$ and the proof is complete.

We close this section with some remarks concerning radicals. Recall that the Jacobson radical $J(M)$ of a module M is equal to the intersection of all maximal submodules. We define D-Jacobson radical $J^D(M)$ of a module M with a d-derivation D as the intersection of all maximal D-submodules. As a consequence of 3.1(a) we obtain the following

3.4. Corollary. *If the d-derivation D has index v, then $J(R)^{v-1} \cdot J(M) \subseteq J^D(M)$.*

Proof. Let N be a maximal D-submodule of M. Obviously $\tilde{D}$-$l(M/N) = 1$ and by 3.1(a) there exists a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = N$$

of submodules of M such that all factors M_i/M_{i+1} are simple modules and $n \leqq v$. Now a simple induction on $j \leqq n$ gives that $J(R)^{j-1} \cdot J(M) \subseteq M_j$. In particular $J(R)^{v-1} \cdot J(M) \subseteq J(R)^{n-1} J(M) \subseteq M_n = N$. Consequently $J(R)^{v-1} \cdot J(M) \subseteq J^D(M)$.

The following example shows that in general the exponent $v - 1$ in 3.4 cannot be improved.

3.5. Example. Let $T = Z_p[x]$ be the polynomial ring in the indeterminate x over the field Z_p, and let $I = (x)$ denote its augumentation ideal. The derivation on T defined by setting $d(x) = 1$ maps the ideal I^p into itself; so d induces derivation $\bar{d}$ on $\bar{T} = T/I^p$. Obviously $\bar{d}$ has index p, $J^{\bar{d}}(\bar{T}) = 0$ and $J(\bar{T}) = I/I^p$.

4. Clifford type relations

The aim of this final section is, among other things, to study D-simple modules.

We begin with two elementary lemmas.

4.1. Lemma. *Let D be a derivation on a left R-module M. If for some $r \in R$ and $m \in M$, $0 = rm = rD(m) = \cdots = rD^{k-1}(m)$ then $d^{k-1}(r)m = \cdots = d(r)m = rm = 0$ and $rD^k(m) = (-1)^k d^k(r)m$.*

Proof. We prove by induction on s that

$$(*) \qquad d^s(r)D^l(m) = 0 \qquad \text{for } 0 \leqq s + l \leqq k - 1.$$

The case $s = 0$ is clear. Now let $0 \leqq s < k - 1$ and let $s + l + 1 \leqq k - 1$. The induction assumption gives that $d^s(r)D^l(m) = d^s(r)D^{l+1}(m) = 0$. Hence

$$0 = D(d^s(r)D^l(m)) = d^{s+1}(r)D^l(m).$$

This proves (*). In particular (*) implies that $d^{k-1}(r)m = \cdots = d(r)m = rm = 0$. Moreover it gives that for every $s = 0, 1, \ldots, k-1$,

$$0 = D(d^s(r)D^{k-s-1}(m)) = d^{s+1}(r)D^{k-(s+1)}(m) + d^s(r)D^{k-s}(m).$$

Therefore $rD^k(m) = -d(r)D^{k-1}(m) = \cdots = (-1)^k d^k(r)m$.

4.2. Lemma. $\det V_n = n + 1$, *where*

$$V_n = \begin{pmatrix} \binom{n+1}{1} & \binom{n+1}{2} & \cdots & \binom{n+1}{n} \\ \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{n-1} \\ 0 & \binom{n-1}{0} & \cdots & \binom{n-1}{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \binom{2}{1} \end{pmatrix}$$

Proof. Consider the $n \times n$ matrix

$$W_n = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \in M_n(\mathbf{Z}).$$

It is clear that the determinant is invariant under elementary operations, i.e., operations which add to a row of a matrix a linear combination of its rows. By induction on n we prove that using elementary operations we can pass from V_n to W_n. It is clear for $n = 1$. Now let $n > 1$. By induction we can pass from V_n to

$$W_n' = \begin{pmatrix} \binom{n+1}{1} & \binom{n+1}{2} & \cdots & \binom{n+1}{n} \\ 1 & & & \\ 0 & & W_{n-1} & \\ \vdots & & & \\ 0 & & & \end{pmatrix}.$$

If w_j denotes the j-th row of W_n', then it is easy to check that

$$(2,1,0,\dots,0)+\binom{n-1}{1}w_2+\binom{n-1}{2}w_3+\cdots+\binom{n-1}{n-1}w_n = w_1,$$

and hence we can pass from V_n to W_n.

Therefore $\det V_n = \det W_n = 2\det W_{n-1} - \det W_{n-2}$. Since $\det W_1 = 2$ and $\det W_2 = 3$, we obtain that $\det V_n = n+1$.

4.3. Proposition. *Suppose that d is a derivation on R of finite index w and that $w!$ is invertible in R. Then for every left R-module M with d-derivation D and every essential submodule N of M, the submodule $b(N) = N \cap D^{-1}(N)$ is also essential in M.*

Proof. Suppose that $b(N)$ is not essential. Using essentiality of N we can choose $x \in N$ such that $b(N) \cap Rx = 0$ and $r \in R$ with $0 \neq rS \subseteq N$, where $S = \{x, D(x), \dots, D^w(x)\}$.

Let k $(0 \leqq k \leqq w)$ be minimal with respect to $rD^k(x) \neq 0$. We claim that $0 < k < w$. Indeed, if $rx \neq 0$ then $rD(x) \in N$ and $D(rx) = d(r)x + rD(x) \in N$. Hence $rx \in b(N) \cap Rx = 0$, a contradiction. On the other hand if $k = w$ then, by 4.1 and 1.3, $rD^w(x) = (-1)^w d^w(r)x \in \Sigma_{j=0}^{w-1} Rd^j(r)x = 0$ which is impossible. Thus $0 < k < w$ and

$$0 = rx = rD(x) = \cdots = rD^{k-1}(x).$$

By Leibniz formula 1.1 we obtain

$$0 = D^{k+1}(rx) = d^{k+1}(r)x + \sum_{i=1}^{k}\binom{k+1}{i} d^{k+1-i}(r)D^i(x) + rD^{k+1}(x).$$

Moreover for $s = 2,\dots,k$ we have

$$0 = D^s(rD^{k-s+1}(x)) = \sum_{i=0}^{s-1}\binom{s}{i} d^{s-i}(r)D^{k-s+1+i}(x) + rD^{k+1}(x).$$

Now putting $x_j = d^{k+1-j}(r)D^j(x)$ $(j = 1,\dots,k)$ we obtain

$$\sum_{j=1}^{k}\binom{k+1}{j} x_j \in N \quad\text{and}\quad \sum_{j=s}^{k}\binom{k-s+1}{j-s} x_j \in N, \qquad \text{for } s = 1,\dots,k-1.$$

By 4.2 det $V_k = k+1 \leqq w$ is invertible in R. Hence for $j = 1, \ldots, k$, $x_j \in N$ and in particular $d^k(r)D(x) \in N$. We have $D(d^k(r)x) = d^{k+1}(r)x + d^k(r)D(x) \in N$ and, by 4.1, $0 \neq d^k(r)x \in b(N) \cap Rx$. This contradiction ends the proof.

The above result clarifies the following

4.4. Corollary. *Under the assumption on d of 4.3 we have*

(a) *for every essential left ideal I of R, the left ideal I_* is also essential*;

(b) *for every left R-module M with d-derivation D of finite index and each essential submodule N of M, N_* is also an essential submodule of M.*

As a final result we prove the following

4.5. Theorem. *Let d be a derivation on R of finite index w with $w!$ invertible in R and let D be a d-derivation of finite index v on a left R-module M.*

(a) *If M is D-simple, then M is a completely reducible module of length $l(M) \leqq v$.*

(b) $J(M) \subseteq J^D(M)$.

(c) *If the ring R has no proper d-invariant left ideals then R is a simple Artinian ring. If in addition R is a commutative ring then R must be a field.*

Proof. (a) It is an immediate consequence of 4.4(b) and 3.1(a).

(b) It follows from 4.5(a).

(c) By (a) and (b) R is a semisimple Artinian ring. Hence, it suffices to show that R is a simple ring. Suppose that I is a non-zero ideal of R. Then there exists a central idempotent $e \in I$ such that $I = Ie$. Obviously $d(I) = d(Ie) = d(I)e + Id(e) \subseteq I$. Therefore $I = R$ and R is a simple ring.

The following example points out differences between properties of $J(R)$ and $J^d(R)$.

4.6. Example. Let $R = M_2(Q)$ be the full 2×2 matrix ring over the field of rational numbers, and let d_a be the inner derivation on R determined by the matrix

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Clearly $d_a^3 = 0$, so d_a has index $\leqq 3$. It is easy to see that

$$I = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| \, x, y \in Q \right\}$$

is the unique proper left d_a-ideal of R. Therefore $I = J^{d_a}(R) \neq J(R) = 0$. Moreover $I^2 = I$.

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